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WHEN IS THE INSERTION OF THE GENERATORS INJECTIVE FOR A SUR-REFLECTIVE SUBCATEGORY OF A CATEGORY OF MANY-SORTED ALGEBRAS?

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Abstract. For a many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ we characterize, by defining the concept of support of an S-sorted set and a convenient algebraic closure operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ on S, those sur-reflective subcategories $\boldsymbol{\mathcal{K}}$ of the category $\mathbf{Alg}(\mathbf{\Sigma})$ of all $\mathbf{\Sigma}$-algebras for which the unit of the adjunction from $\mathbf{Set}^S$ to $\boldsymbol{\mathcal{K}}$ is pointwise monomorphic.

1. Introduction.

Let Σ be a single-sorted signature and $\mathcal{K}$ a set of Σ-algebras abstract (i.e., closed under isomorphic algebras) and closed under subalgebras and direct products or, what is equivalent (see, e.g., [9], Theorem 13, p. 197), a sur-reflective subcategory $\boldsymbol{\mathcal{K}}$ of the category $\mathbf{Alg}(\Sigma)$ of all Σ-algebras. Then the non-triviality of $\mathcal{K}$, as it is well known, is a necessary and sufficient condition for the pointwise injectivity of the unit ν of $\mathbf{T}_{\Sigma,\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}} \colon \mathbf{Set} \rightharpoonup \boldsymbol{\mathcal{K}}$, the canonical adjunction from $\mathbf{Set}$ to $\boldsymbol{\mathcal{K}}$, where $\mathbf{Set}$ is the category of sets, $\boldsymbol{\mathcal{K}}$ the sur-reflective subcategory of $\mathbf{Alg}(\Sigma)$ determined by $\mathcal{K}$, $G_{\boldsymbol{\mathcal{K}}}$ the forgetful functor from $\boldsymbol{\mathcal{K}}$ to $\mathbf{Set}$, and $\mathbf{T}_{\Sigma,\boldsymbol{\mathcal{K}}}$ the functor from $\mathbf{Set}$ to $\boldsymbol{\mathcal{K}}$ which assigns to a set X the free $\boldsymbol{\mathcal{K}}$-algebra over X. We notice that we take $\mathbf{Set}$ to be the category whose set of objects is $\boldsymbol{\mathcal{U}}$, a Grothendieck universe fixed once and for all. Moreover, for every Σ-algebra $\mathbf{A} = (A, F)$, we assume that $A \in \boldsymbol{\mathcal{U}}$.

The corresponding situation in the many-sorted case is somewhat different. Firstly, we recall that, for a many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ is called *subfinal* if, for every $\mathbf{\Sigma}$-algebra $\mathbf{B}$, there exists at most one $\mathbf{\Sigma}$-homomorphism

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from $\mathbf{B}$ to $\mathbf{A}$. Let $\mathbf{\Sigma} = (S, \Sigma)$ be a many-sorted signature and $\mathcal{K}$ a set of $\mathbf{\Sigma}$-algebras abstract and closed under subalgebras and direct products or, what is equivalent (see [8], Corollar 3.9, p. 29), a sur-reflective subcategory $\boldsymbol{\mathcal{K}}$ of the category $\mathbf{Alg}(\mathbf{\Sigma})$ of all $\mathbf{\Sigma}$-algebras. Then the *non-triviality* of $\mathcal{K}$, that is, the condition that $\mathcal{K}$ contains at least one non-subfinal $\mathbf{\Sigma}$-algebra, although necessary is not sufficient, in general, for the pointwise injectivity of the unit ν of $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}} \colon \mathbf{Set}^S \longrightarrow \boldsymbol{\mathcal{K}}$, the canonical adjunction from $\mathbf{Set}^S$ to $\boldsymbol{\mathcal{K}}$, where $\mathbf{Set}^S$ is the usual functor category and $\boldsymbol{\mathcal{K}}$, $G_{\boldsymbol{\mathcal{K}}}$, and $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}$ are the many-sorted counterparts of its respective single-sorted homologous. From now on, for a $\mathbf{\Sigma}$-algebra $\mathbf{A} = (A, F)$, it is assumed that $A = (A_s)_{s\in S} \in \boldsymbol{\mathcal{U}}^S$. Moreover, by abuse of notation, we let $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ or $\mathbf{A} \in \mathbf{Alg}(\mathbf{\Sigma})$ stands for $\mathbf{A} \in \mathcal{K}$ or $\mathbf{A} \in \mathrm{Alg}(\mathbf{\Sigma})$, respectively, where $\mathcal{K}$ and $\mathrm{Alg}(\mathbf{\Sigma})$ denote, respectively, the sets of objects of the categories $\boldsymbol{\mathcal{K}}$ and $\mathbf{Alg}(\mathbf{\Sigma})$.

There are examples of many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, with $\mathrm{card}(S) \geq 2$, and of sur-reflective subcategories $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\mathbf{\Sigma})$ for which the non-triviality of $\boldsymbol{\mathcal{K}}$ is a sufficient condition for the pointwise injectivity of the unit of $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}}$ (and consequently equivalent to it, since the non-triviality of $\boldsymbol{\mathcal{K}}$ is always a necessary condition for the pointwise injectivity of the unit of $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}}$). For instance, one may form the category $\mathbf{Act}$ of all left acts over all monoids. An object of $\mathbf{Act}$ is a pair $(\mathbf{M}, X)$, where $\mathbf{M}$ is a monoid and X a left $\mathbf{M}$-act (see [6], p. 43). A morphism $(\mathbf{M}, X) \longrightarrow (\mathbf{M}', X')$ is a pair (f, g), where $f \colon \mathbf{M} \longrightarrow \mathbf{M}'$ is a morphism of monoids and $g \colon X \longrightarrow X'$ is a morphism of left $\mathbf{M}$-acts, that is, for every $m \in M$ and every $x \in X$, $g(m \cdot x) = f(m) \cdot g(x)$. Given the forgetful functor $G_{\mathbf{Act}} \colon \mathbf{Act} \longrightarrow \mathbf{Set}^2$ and a 2-sorted set (Y, X), we have that $\mathbf{T}_{\mathbf{Act}}(Y, X)$, the free left act over (Y, X), is $(\mathbf{Y}^\star, Y^\star \times X)$, where $\mathbf{Y}^\star$ is the free monoid over Y and $Y^\star$ its underlying set, and that the 2-sorted mapping from (Y, X) to $(\mathbf{Y}, Y^\star \times X)$ which sends (y, x) to $((y), (\lambda, x))$, with λ the empty word over Y and (y) the word of length one over Y associated to y, is an embedding. Therefore the unit of the adjunction $\mathbf{T}_{\mathbf{Act}} \dashv G_{\mathbf{Act}} \colon \mathbf{Set}^2 \longrightarrow \mathbf{Act}$ is pointwise injective.

For another example of the same genre as the above consider the category $\mathbf{Mod}$ of all left modules over all rings (see [7], p. 35).

On the other hand, there are also examples of many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, with $\mathrm{card}(S) \geq 2$, and of sur-reflective subcategories $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\mathbf{\Sigma})$ for which the non-triviality of $\boldsymbol{\mathcal{K}}$ is not a sufficient condition for the pointwise injectivity of the unit of $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}}$. For instance, let $\mathbf{\Sigma}$ be the many-sorted signature with sort set $S = \{s_0, s_1\}$, and with $\Sigma_{w,s} = \{\sigma\}$, if $w = \lambda$ and $s = s_0$, and $\Sigma_{w,s} = \varnothing$ for all other w, s, and let $\mathcal{K}$ be the set of all $\mathbf{\Sigma}$-algebras $\mathbf{A}$ such that $\mathrm{card}(A_{s_1}) \leq 1$. Then $\mathcal{K}$ is a set of $\mathbf{\Sigma}$-algebras non-trivial, abstract, and closed under subalgebras and direct products. Moreover, for the S-sorted set $(\varnothing, 2)$, the canonical S-sorted mapping from $(\varnothing, 2)$ to $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(\varnothing, 2) = G_{\boldsymbol{\mathcal{K}}}(\mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(\varnothing, 2))$ is not injective. We recall that $\mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(\varnothing, 2)$ is the quotient of $\mathbf{T}_{\boldsymbol{\Sigma}}(\varnothing, 2)$, the absolutely free $\boldsymbol{\Sigma}$-algebra over $(\varnothing, 2)$, by $\equiv^{\boldsymbol{\mathcal{K}}}_{\mathbf{T}_{\boldsymbol{\Sigma}}(\varnothing,2)}$, the congruence on $\mathbf{T}_{\boldsymbol{\Sigma}}(\varnothing, 2)$ defined as

$$\equiv^{\boldsymbol{\mathcal{K}}}_{\mathbf{T}_{\boldsymbol{\Sigma}}(\varnothing,2)} = \bigcap_{\substack{f \in \mathrm{Hom}(\mathbf{T}_{\boldsymbol{\Sigma}}(\varnothing,2),\mathbf{A}) \\ \&\ \mathbf{A} \in \boldsymbol{\mathcal{K}}}} \mathrm{Ker}(f).$$

Another example which falls within the same genre as the just stated is obtained as follows. Let $\boldsymbol{\Sigma}$ be the many-sorted signature with sort set $S = \{s_0, s_1\}$, and with $\Sigma_{w,s} = \{\sigma\}$, if $w = (s_0)$, the word of length one over S associated to s_0, and $s = s_1$, and $\Sigma_{w,s} = \varnothing$ for all other w, s, and let $\mathcal{K}$ be the set of all $\boldsymbol{\Sigma}$-algebras $\mathbf{A}$ such that $\mathrm{card}(A_{s_0}) \leq 1$. Then $\mathcal{K}$ is a set of $\boldsymbol{\Sigma}$-algebras non-trivial, abstract, and closed under subalgebras and direct products. Moreover, for the S-sorted set $(2, \varnothing)$, the canonical S-sorted mapping from $(2, \varnothing)$ to $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(2, \varnothing)$ is not injective.

An additional example related to parameterized data type definitions can be easily extracted from that one provided in [4], p. 311.

The main goal of this article is to obtain, for an arbitrary but fixed many-sorted signature $\boldsymbol{\Sigma} = (S, \Sigma)$, a characterization of those sur-reflective subcategories $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\boldsymbol{\Sigma})$ for which it happens that, for every S-sorted set X, the insertion ν^X of the generators X into $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X) = G_{\boldsymbol{\mathcal{K}}}(\mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X))$ is injective. The method we have used to attain this goal is, ultimately, founded on the definition of the concept of support mapping for S and on the definition of a suitable algebraic closure operator on S.

We next proceed to describe the contents of the second, and final, section of this article. In it we begin by defining, for every set of sorts S, the support mapping for S, denoted by supp_S, as the mapping from $\boldsymbol{\mathcal{U}}^S$, the set of all S-sorted sets, to $\mathrm{Sub}(S)$, the set of all subsets of S, which assigns to $A = (A_s)_{s\in S} \in \boldsymbol{\mathcal{U}}^S$ its support $\mathrm{supp}_S(A) = \{\, s \in S \mid A_s \neq \varnothing \,\}$. Then, given a sort $t \in S$, we continue by defining the Kronecker delta δ^t at t, which is a special S-sorted set. These Kronecker deltas will be used to define some S-sorted sets which are necessary to set up some proofs. We notice in passing that the Kronecker deltas δ^t, for $t \in S$, are the building blocks of the S-sorted sets, since, for every S-sorted set A, we have that A is isomorphic to $\coprod_{s\in S} \mathrm{card}(A_s) \cdot \delta^s$, where $\mathrm{card}(A_s) \cdot \delta^s = \coprod_{\alpha\in \mathrm{card}(A_s)} \delta^s$, for all $s \in S$.

After having done that we assign to every many-sorted signature $\boldsymbol{\Sigma} = (S, \Sigma)$ an algebraic closure operator $\mathrm{Ex}_{\boldsymbol{\Sigma}}$ on S. The algebraic closure operator $\mathrm{Ex}_{\boldsymbol{\Sigma}}$ sends a subset T of S to $\mathrm{Ex}_{\boldsymbol{\Sigma}}(T) = \mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma}}(X))$, its $\boldsymbol{\Sigma}$-extent, where X is any S-sorted set such that $\mathrm{supp}_S(X) = T$ and $\mathrm{T}_{\mathbf{\Sigma}}(X)$ the underlying S-sorted set of $\mathbf{T}_{\mathbf{\Sigma}}(X)$, the absolutely free $\mathbf{\Sigma}$-algebra over X.

Following this, for every many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, we prove that the closed sets of the algebraic closure system associated to $\mathrm{Ex}_{\mathbf{\Sigma}}$ are exactly the supports of the underlying S-sorted sets of all $\mathbf{\Sigma}$-algebras. We next prove that, for a set of $\mathbf{\Sigma}$-algebras $\mathcal{K}$ abstract and closed under subalgebras and direct products, the mappings supp_S, $\mathrm{Ex}_{\mathbf{\Sigma}}$, and $\mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}$, where $\mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}$ stands for the object mapping of the functor $G_{\boldsymbol{\mathcal{K}}} \circ \mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}$, are such that $\mathrm{supp}_S \circ \mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} = \mathrm{Ex}_{\mathbf{\Sigma}} \circ \mathrm{supp}_S$.

After having stated all of these auxiliary results we provide a solution to the problem posed in the title of this paper. Concretely, we prove the following characterization theorem: For a many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ and a sur-reflective subcategory $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\mathbf{\Sigma})$, the unit ν of the adjunction $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}} \colon \mathbf{Set}^S \longrightarrow \boldsymbol{\mathcal{K}}$ is pointwise monomorphic if and only if there are "enough" $\mathbf{\Sigma}$-algebras in $\boldsymbol{\mathcal{K}}$. In other words, we prove, on account of the above alternative, but equivalent, description of the fixed points of $\mathrm{Ex}_{\mathbf{\Sigma}}$, that ν is pointwise monomorphic if and only if, for every $\mathbf{\Sigma}$-algebra $\mathbf{B}$ and for every sort s in $\mathrm{supp}_S(B)$, there exists at least one $\mathbf{\Sigma}$-algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{K}}$ such that $\mathrm{supp}_S(A) = \mathrm{supp}_S(B)$ and $\mathrm{card}(A_s) \geq 2$.

To finish this introductory section it is appropriate to remark that there is a connection between the just mentioned characterization theorem and a theorem stated by Birkhoff-Frink in [1]. Specifically, in [1], p. 300, it is proved that any algebraic closure space arises as the subalgebra closure space for some algebra structure on the underlying set of the algebraic closure space. More accurately, it is proved that, for every algebraic closure space (A, J), there exists a single-sorted signature $\Sigma^{(A,J)}$ and a $\Sigma^{(A,J)}$-algebra structure $F^{(A,J)}$ on A such that $(A, \mathrm{Sg}_{(A,F^{(A,J)})}) = (A, J)$, where $\mathrm{Sg}_{(A,F^{(A,J)})}$ is the subalgebra generating operator on A induced by the $\Sigma^{(A,J)}$-algebra $(A, F^{(A,J)})$. Therefore we can assert that, for a many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ and a sur-reflective subcategory $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\mathbf{\Sigma})$, the characterization of the pointwise injectivity of the unit ν of the adjunction from $\mathbf{Set}^S$ to $\boldsymbol{\mathcal{K}}$, keeping in mind the aforementioned theorem of Birkhoff-Frink, has been partially done in terms of a single-sorted algebra $(S, F^{(S,\mathrm{Ex}_{\mathbf{\Sigma}})})$ (and of the concept of support mapping for S).

In all that follows we use standard concepts and constructions from category theory, see [2] and [7], and from many-sorted algebra, see [8] and [9].

2. The characterization theorem.

Our main goal in this section is to state a theorem which characterizes those sur-reflective subcategories of a category of many-sorted algebras for which the values of the units of the corresponding adjunctions are injective. However, to attain the goal just mentioned we need to begin by defining, for a set of sorts S, the concept of support of an S-sorted set and that of Kronecker delta at a sort in the set of sorts S.

Definition 2.1. Let S be a set of sorts in $\boldsymbol{\mathcal{U}}$.

(1) The *support of an S-sorted set* $A = (A_s)_{s\in S} \in \boldsymbol{\mathcal{U}}^S$, denoted by $\mathrm{supp}_S(A)$, is $\{\, s \in S \mid A_s \neq \varnothing \,\}$. From now on, supp_S stands for the mapping from $\boldsymbol{\mathcal{U}}^S$ to $\mathrm{Sub}(S)$ which sends an S-sorted set to its support, and we call it the *support mapping for S*.

(2) Let t be a sort in S. Then the *Kronecker delta at t*, denoted by δ^t, is the S-sorted set $(\delta^t_s)_{s\in S}$ defined, for every $s \in S$, as follows: $\delta^t_s = 1$, if $s = t$, and $\delta^t_s = \varnothing$, otherwise.

Remark. The concept of support does not play any significant role in the case of the single-sorted algebras. Nevertheless, it has turned to be essential to accomplish some investigations in the field of many-sorted algebras, for example, that one in [3]. Moreover, the family $\mathrm{supp} = (\mathrm{supp}_S)_{S\in\boldsymbol{\mathcal{U}}}$ is a pseudo-natural-transformation from a pseudo-functor to a functor, both from the category **Set** to the 2-category **Cat**.

In the following proposition, for a set of sorts S, we gather together only those properties of the support mapping for S which will actually be used in the proofs of some of the propositions following it.

Proposition 2.2. *Let S be a set of sorts, A, B two S-sorted sets, Φ an S-sorted equivalence relation on A, that is, $\Phi = (\Phi_s)_{s\in S}$, where, for every $s \in S$, Φ_s is an equivalence relation on A_s, and $(A^i)_{i\in I} = ((A^i_s)_{s\in S})_{i\in I}$ a family of S-sorted sets indexed by a set I. Then the following properties hold:*

(1) $\mathrm{Hom}(A, B) \neq \varnothing$ *iff* $\mathrm{supp}_S(A) \subseteq \mathrm{supp}_S(B)$. *Therefore, if $A \subseteq B$, that is, if, for every $s \in S$, $A_s \subseteq B_s$, then* $\mathrm{supp}_S(A) \subseteq \mathrm{supp}_S(B)$.

(2) *If from A to B there exists a surjective S-sorted mapping, then we have that* $\mathrm{supp}_S(A) = \mathrm{supp}_S(B)$. *Therefore,* $\mathrm{supp}_S(A) = \mathrm{supp}_S(A/\Phi)$, *where, for every $s \in S$, $(A/\Phi)_s = A_s/\Phi_s$.*

(3) $\mathrm{supp}_S(\bigcup_{i\in I} A^i) = \bigcup_{i\in I} \mathrm{supp}_S(A^i)$, *where, for every $s \in S$, $(\bigcup_{i\in I} A^i)_s$ is $\bigcup_{i\in I} A^i_s$.*

(4) *If $I \neq \varnothing$, then* $\mathrm{supp}_S(\prod_{i\in I} A^i) = \bigcap_{i\in I} \mathrm{supp}_S(A^i)$, *where, for every $s \in S$, $(\bigcap_{i\in I} A^i)_s$ is $\bigcap_{i\in I} A^i_s$ and $(\prod_{i\in I} A^i)_s$ is $\prod_{i\in I} A^i_s$.*

Proof. The proof is straightforward. □

Our next objective is to assign, *in a natural way*, to every many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ an algebraic closure operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ on S, and to state and prove those facts about the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ which will used afterwards in the proof of the characterization theorem.

Proposition 2.3. *Let $\mathbf{\Sigma} = (S, \Sigma)$ be a many-sorted signature. Then the self-mapping $\mathrm{Ex}_{\mathbf{\Sigma}}$ of $\mathrm{Sub}(S)$ which sends a subset T of S to $\mathrm{Ex}_{\mathbf{\Sigma}}(T) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X))$, where X is any S-sorted set such that $\mathrm{supp}_S(X) = T$, for example, $X = \bigcup_{s\in T} \delta^s$, is an algebraic closure operator on S. We agree to call the value of $\mathrm{Ex}_{\mathbf{\Sigma}}$ at a subset T of S the* $\mathbf{\Sigma}$-extent of T.

Proof. The self-mapping $\mathrm{Ex}_{\mathbf{\Sigma}}$ is well-defined, i.e., if Y is another S-sorted set such that also $\mathrm{supp}_S(Y) = T$, then $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X)) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(Y))$. As a matter of fact, from $\mathrm{supp}_S(Y) = T$, we deduce, since, by hypothesis, $\mathrm{supp}_S(X) = T$, that $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Y, X)$ are nonempty, thus neither $\mathrm{Hom}(\mathbf{T}_{\mathbf{\Sigma}}(X), \mathbf{T}_{\mathbf{\Sigma}}(Y))$ nor $\mathrm{Hom}(\mathbf{T}_{\mathbf{\Sigma}}(Y), \mathbf{T}_{\mathbf{\Sigma}}(X))$ is empty, therefore, by the first part of Proposition 2.2, $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X)) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(Y))$.

Having checked that the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ is well-defined, we continue by proving that it is an algebraic closure operator on S.

Let us first prove that the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ is extensive. Let $T \subseteq S$ be and $t \in T$. Then, since $\mathrm{supp}_S(X) = T$, $t \in \mathrm{supp}_S(X)$, that is, $X_t \neq \varnothing$. But from X to $\mathrm{T}_{\mathbf{\Sigma}}(X)$ we have the S-sorted mapping η^X, that is, the value at X of the unit η of the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv G_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$, thus $\mathrm{T}_{\mathbf{\Sigma}}(X)_t \neq \varnothing$, that is, $t \in \mathrm{Ex}_{\mathbf{\Sigma}}(T)$. Therefore $\mathrm{Ex}_{\mathbf{\Sigma}}$ is extensive.

Our next objective is to prove that the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ is isotone. Let T, $U \subseteq S$ be such that, $T \subseteq U$. Then taking an S-sorted set X such that $\mathrm{supp}_S(X) = T$ and another S-sorted set Y such that $\mathrm{supp}_S(Y) = U$, since there exists an S-sorted mapping from X to Y, there exists also a $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Sigma}}(Y)$. Thus, if $s \in \mathrm{Ex}_{\mathbf{\Sigma}}(T)$, that is, if $\mathrm{T}_{\mathbf{\Sigma}}(X)_s \neq \varnothing$, then also $\mathrm{T}_{\mathbf{\Sigma}}(Y)_s \neq \varnothing$, that is, $s \in \mathrm{Ex}_{\mathbf{\Sigma}}(U)$. Therefore $\mathrm{Ex}_{\mathbf{\Sigma}}$ is isotone.

We next prove that the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ is idempotent. It is obvious that, for a subset T of S, $\mathrm{Ex}_{\mathbf{\Sigma}}(T) \subseteq \mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{Ex}_{\mathbf{\Sigma}}(T))$. Reciprocally, if $s \in \mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{Ex}_{\mathbf{\Sigma}}(T))$, then $s \in \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(Y))$, for some S-sorted set Y such that $\mathrm{supp}_S(Y) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X))$, where X is an S-sorted set such that $\mathrm{supp}_S(X) = T$. Moreover, from the equality $\mathrm{supp}_S(Y) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X))$ it follows, by the first part of Proposition 2.2, that $\mathrm{Hom}(Y, \mathrm{T}_{\mathbf{\Sigma}}(X)) \neq \varnothing$, thus $\mathrm{Hom}(\mathbf{T}_{\mathbf{\Sigma}}(Y), \mathbf{T}_{\mathbf{\Sigma}}(X)) \neq \varnothing$. But $\mathrm{T}_{\mathbf{\Sigma}}(Y)_s \neq \varnothing$, therefore $\mathrm{T}_{\mathbf{\Sigma}}(X)_s \neq \varnothing$, that is, $s \in \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X))$, hence $s \in \mathrm{Ex}_{\mathbf{\Sigma}}(T)$. From this we can assert that $\mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{Ex}_{\mathbf{\Sigma}}(T)) \subseteq \mathrm{Ex}_{\mathbf{\Sigma}}(T)$. Thus $\mathrm{Ex}_{\mathbf{\Sigma}}$ is idempotent.

Finally, we prove that the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ is algebraic, that is, that, for every subset T of S, the following equality $\mathrm{Ex}_{\mathbf{\Sigma}}(T) = \bigcup_{U\in\mathrm{Sub_f}(T)} \mathrm{Ex}_{\mathbf{\Sigma}}(U)$ holds, where $\mathrm{Sub_f}(T)$ is the set of all finite subsets of T. Let T be a subset of S. Then $\mathrm{Ex}_{\mathbf{\Sigma}}(T)$ can be represented, for example, as $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(\bigcup_{s\in T}\delta^s))$. Moreover, taking into account that, for every S-sorted set X, $\mathrm{T}_{\mathbf{\Sigma}}(X)$ is essentially $\mathrm{Sg}_{\mathbf{W}_{\mathbf{\Sigma}}(X)}(X)$, the subalgebra of the $\mathbf{\Sigma}$-algebra $\mathbf{W}_{\mathbf{\Sigma}}(X)$ of $\mathbf{\Sigma}$-rows in X generated by X, and that the operator $\mathrm{Sg}_{\mathbf{W}_{\mathbf{\Sigma}}(X)}$ is, in particular, algebraic, we can affirm that $\mathrm{T}_{\mathbf{\Sigma}}(\bigcup_{s\in T}\delta^s)$ is isomorphic to $\bigcup_{U\in\mathrm{Sub_f}(T)}\mathrm{T}_{\mathbf{\Sigma}}(\bigcup_{s\in U}\delta^s)$. Therefore, by the third part of Proposition 2.2 and since two isomorphic S-sorted sets have, obviously, the same support, we can affirm that $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(\bigcup_{s\in T}\delta^s)) = \bigcup_{U\in\mathrm{Sub_f}(T)}(\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(\bigcup_{s\in U}\delta^s)))$, hence, as we wanted to prove, $\mathrm{Ex}_{\mathbf{\Sigma}}(T) = \bigcup_{U\in\mathrm{Sub_f}(T)}\mathrm{Ex}_{\mathbf{\Sigma}}(U)$. Thus $\mathrm{Ex}_{\mathbf{\Sigma}}$ is algebraic. This completes the proof. □

Remark. The family $(\mathrm{Ex}_{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}}$ is actually the object mapping of a functor Ex from a convenient category **Sig** of many-sorted signatures to the category **AClSp** of algebraic closure spaces and continuous mappings. This shows that the assignation we have stated in Proposition 2.3 is definitely a natural one.

From Proposition 2.3 it follows that $\mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}}) = \{\, T\subseteq S \mid T = \mathrm{Ex}_{\mathbf{\Sigma}}(T)\,\}$, the set of all fixed points of the algebraic closure operator $\mathrm{Ex}_{\mathbf{\Sigma}}$, is an algebraic closure system on S. On the other hand, since the elements of $\mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}})$, on account of the definition of the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$, have a complicated description, we provide in the following proposition a more tractable characterization of them based on the supports of the underlying S-sorted sets of all $\mathbf{\Sigma}$-algebras in $\mathbf{Alg}(\mathbf{\Sigma})$.

Proposition 2.4. *Let $\mathbf{\Sigma} = (S,\Sigma)$ be a many-sorted signature. Then we have that*

$$\mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}}) = \{\, \mathrm{supp}_S(A) \mid \mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})\,\}.$$

PROOF. Let us first prove that $\mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}}) \subseteq \{\, \mathrm{supp}_S(A) \mid \mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})\,\}$. Indeed, if $T\in\mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}})$, then, for an $S-$sorted set X such that $\mathrm{supp}_S(X) = T$, we have that $T = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(X))$. Thus $T\in\{\, \mathrm{supp}_S(A) \mid \mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})\,\}$.

Reciprocally, if $\mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})$, then we have that $\mathrm{supp}_S(A)\subseteq\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(A))$, because there is an S-sorted mapping from A to $\mathrm{T}_{\mathbf{\Sigma}}(A)$, precisely η^A, the insertion of A into $\mathrm{T}_{\mathbf{\Sigma}}(A)$, and that $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(A))\subseteq\mathrm{supp}_S(A)$, because from $\mathbf{T}_{\mathbf{\Sigma}}(A)$ to $\mathbf{A}$ there exists at least one $\mathbf{\Sigma}$-homomorphism, for example, the canonical extension $\mathrm{id}_A^\sharp$ of id_A to $\mathbf{T}_{\mathbf{\Sigma}}(A)$. Therefore $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(A)) = \mathrm{supp}_S(A)$. Hence, from this equality and the definition of the operator $\mathrm{Ex}_{\mathbf{\Sigma}}$ in Proposition 2.3, it may be concluded that

$$\mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{supp}_S(A)) = \mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma}}(A)) = \mathrm{supp}_S(A),$$

that is, that $\mathrm{supp}_S(A)$ is a fixed point of $\mathrm{Ex}_{\boldsymbol{\Sigma}}$. This completes the proof. $\square$

The proposition just stated can also be interpreted, dually, as meaning that, for a many-sorted signature $\boldsymbol{\Sigma} = (S, \Sigma)$, the set obtained by gathering together the supports of the underlying S-sorted sets of all $\boldsymbol{\Sigma}$-algebras in $\mathbf{Alg}(\boldsymbol{\Sigma})$ is not an amorphous set, but an algebraic closure system on S, and that its canonically associated algebraic closure operator on S is precisely $\mathrm{Ex}_{\boldsymbol{\Sigma}}$.

Before stating the relation between the mappings supp_S, $\mathrm{Ex}_{\boldsymbol{\Sigma}}$, and the object mapping of the functor $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}} = G_{\boldsymbol{\mathcal{K}}} \circ \mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}$ we provide for the algebraic closure system $\mathrm{Fix}(\mathrm{Ex}_{\boldsymbol{\Sigma}})$ on S, and taking into account Proposition 2.4, the following description of: (1) the top element of $\mathrm{Fix}(\mathrm{Ex}_{\boldsymbol{\Sigma}})$, (2) the meet of a nonempty family in $\mathrm{Fix}(\mathrm{Ex}_{\boldsymbol{\Sigma}})$, and (3) the join of a nonempty directed family in $\mathrm{Fix}(\mathrm{Ex}_{\boldsymbol{\Sigma}})$. The top element is $\mathrm{supp}_S((1)_{s\in S})$, where $(1)_{s\in S}$ is the underlying S-sorted set of the final $\boldsymbol{\Sigma}$-algebra $\mathbf{1}$. For a nonempty family of $\boldsymbol{\Sigma}$-algebras $(\mathbf{A}^i)_{i\in I}$, the meet of $(\mathrm{supp}_S(A^i))_{i\in I}$ is $\mathrm{supp}_S(\prod_{i\in I} A^i)$, since $\prod_{i\in I} \mathbf{A}^i \in \mathbf{Alg}(\boldsymbol{\Sigma})$ and, by the fourth part of Proposition 2.2, we have that $\bigcap_{i\in I} \mathrm{supp}_S(A^i) = \mathrm{supp}_S(\prod_{i\in I} A^i)$. Finally, for a nonempty directed family of $\boldsymbol{\Sigma}$-algebras $(\mathbf{A}^i)_{i\in I}$, the join of $(\mathrm{supp}_S(A^i))_{i\in I}$ is $\bigcup_{i\in I} \mathrm{supp}_S(A^i)$, since $\bigcup_{i\in I} \mathbf{A}^i \in \mathbf{Alg}(\boldsymbol{\Sigma})$ and, by the third part of Proposition 2.2, we have that $\bigcup_{i\in I} \mathrm{supp}_S(A^i) = \mathrm{supp}_S(\bigcup_{i\in I} A^i)$.

For a many-sorted signature $\boldsymbol{\Sigma}$, an S-sorted set X, and a set $\mathcal{K}$ of $\boldsymbol{\Sigma}$-algebras abstract and closed under subalgebras and direct products, we prove in the following proposition that $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X)$ is such that its support equals the $\boldsymbol{\Sigma}$-extent of the support of X which, we warn, contains, in general, strictly the support of X.

Proposition 2.5. *Let $\mathcal{K}$ be a set of $\boldsymbol{\Sigma}$-algebras abstract and closed under subalgebras and direct products. Then the mappings supp_S, $\mathrm{Ex}_{\boldsymbol{\Sigma}}$, and $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}$, where $\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}$ stands for the object mapping of the functor $G_{\boldsymbol{\mathcal{K}}} \circ \mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}$, are such that, for every S-sorted set X, it happens that*

$$\mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X)) = \mathrm{Ex}_{\boldsymbol{\Sigma}}(\mathrm{supp}_S(X)).$$

Proof. From the definition of $\mathrm{Ex}_{\boldsymbol{\Sigma}}$, we have $\mathrm{Ex}_{\boldsymbol{\Sigma}}(\mathrm{supp}_S(X)) = \mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma}}(X))$. Moreover, from $\mathbf{T}_{\boldsymbol{\Sigma}}(X)$ to $\mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X)$ there exists a surjective $\boldsymbol{\Sigma}$-homomorphism, since $\mathbf{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X)$ is a quotient of $\mathbf{T}_{\boldsymbol{\Sigma}}(X)$, thus, by the second part of Proposition 2.2, $\mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma}}(X)) = \mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X))$. Hence $\mathrm{supp}_S(\mathrm{T}_{\boldsymbol{\Sigma},\boldsymbol{\mathcal{K}}}(X)) = \mathrm{Ex}_{\boldsymbol{\Sigma}}(\mathrm{supp}_S(X))$. $\square$

Finally, we state and prove, for a many-sorted signature $\boldsymbol{\Sigma} = (S, \Sigma)$ and on the basis of the support mapping supp_S for S and of the algebraic closure operator $\mathrm{Ex}_{\boldsymbol{\Sigma}}$ on S, the characterization theorem. We notice that this theorem generalizes the classical characterization theorem, mentioned at the beginning of the introductory section, to the context of many-sorted algebras in which, we emphasize, the original theorem fails to apply.

Theorem 2.6. *Let $\mathbf{\Sigma}$ be a many-sorted signature and $\mathcal{K}$ a set of $\mathbf{\Sigma}$-algebras abstract and closed under subalgebras and direct products or, what is equivalent, a sur-reflective subcategory $\boldsymbol{\mathcal{K}}$ of $\mathbf{Alg}(\mathbf{\Sigma})$. Then the following conditions are equivalent:*

1. *For every S-sorted set X, $\nu^X \colon X \longrightarrow \mathrm{T}_{\mathbf{\Sigma},\mathcal{K}}(X)$, the component at X of the unit ν of the adjunction $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}} \dashv G_{\boldsymbol{\mathcal{K}}}$ from $\mathbf{Set}^S$ to $\boldsymbol{\mathcal{K}}$, is injective.*
2. *For every $T \in \mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}})$ and for every sort $s \in T$ there exists at least one $\mathbf{\Sigma}$-algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{K}}$ such that $\mathrm{supp}_S(A) = T$ and $\mathrm{card}(A_s) \geq 2$, or, what is the same, by Proposition 2.4, for every $\mathbf{\Sigma}$-algebra $\mathbf{B}$ and for every $s \in \mathrm{supp}_S(B)$, there exists at least one $\mathbf{\Sigma}$-algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{K}}$ such that $\mathrm{supp}_S(A) = \mathrm{supp}_S(B)$ and $\mathrm{card}(A_s) \geq 2$.*

Proof. Let us assume that, for every S-sorted set X, $\nu^X \colon X \longrightarrow \mathrm{T}_{\mathbf{\Sigma},\mathcal{K}}(X)$ is injective. Let $T \in \mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}})$ be and $s \in T$. Then there exists an S-sorted set X such that $\mathrm{supp}_S(X) = T$ and $\mathrm{card}(X_s) \geq 2$, e.g., $X = \left(\bigcup_{t \in T-\{s\}} \delta^t\right) \cup (\delta^s \coprod \delta^s)$. Hence, for the $\mathbf{\Sigma}$-algebra $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)$, we have, on the one hand, that $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)$ is an element of $\boldsymbol{\mathcal{K}}$, since, by hypothesis, $\mathcal{K}$ is a set of $\mathbf{\Sigma}$-algebras abstract and closed under subalgebras and direct products, on the other hand, that $\mathrm{card}(\mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)_s) \geq 2$, because ν^X_s is injective and $\mathrm{card}(X_s) \geq 2$, and, finally, that $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)) = T$, since $\mathrm{Ex}_{\mathbf{\Sigma}}(T) = T$, $\mathrm{supp}_S(X) = T$, and, by Proposition 2.5, $\mathrm{supp}_S(\mathrm{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)) = \mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{supp}_S(X))$.

Reciprocally, let us assume that, for every $T \in \mathrm{Fix}(\mathrm{Ex}_{\mathbf{\Sigma}})$ and every $s \in T$, there exists at least one $\mathbf{\Sigma}$-algebra $\mathbf{A}$ in $\mathcal{K}$ such that $\mathrm{supp}_S(A) = T$ and $\mathrm{card}(A_s) \geq 2$. Let X be an S-sorted set. Then, for $T = \mathrm{Ex}_{\mathbf{\Sigma}}(\mathrm{supp}_S(X))$ and $s \in T$ there exists at least one $\mathbf{\Sigma}$-algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{K}}$ such that $\mathrm{supp}_S(A) = T$ and $\mathrm{card}(A_s) \geq 2$. Now, given x, $y \in X_s$ such that $x \neq y$ we can assert that there exists an S-sorted mapping $f \colon X \longrightarrow A$ such that $f_s(x) \neq f_s(y)$. Therefore, for the canonical extension $f^\flat$ of f to $\mathbf{T}_{\mathbf{\Sigma},\boldsymbol{\mathcal{K}}}(X)$, $f^\flat_s(\nu^X_s(x)) \neq f^\flat_s(\nu^X_s(y))$, thus $\nu^X_s(x) \neq \nu^X_s(y)$. Hence, ν^X is injective. □

Remark. In [8], the excellent (but, apparently, little known) Dissertation of G. Matthiessen (where he carried out a thorough treatment and deeper discussion of many-sorted universal algebra), the reader will see with dazzling clarity, in opposition to a widespread and unjustified belief, that *many-sorted universal algebra is not at all an inessential variation of classical universal algebra.* To be more precise, the reader will learn when studying [8] that even though the well known semantical characterizations for the single-sorted algebras are transferable to the *finitary quasi-equational* classes and to the *equational* classes of many-sorted algebras this is not so, however, for the *quasi-equational* classes and the *finitary equational* classes of many-sorted algebras (the referee in his report also emphasizes this aspect when he talks about Birkhoff's variety theorem). Notice that, surprisingly, these facts have been simply ignored in the writings devoted to this topic since 1976 onwards or have been partially rediscovered subsequently. In this regard it is worth pointing out that [8] has been cited, as early as 1979, by Grätzer in [5] (the standard reference in classical universal algebra), p. 540.

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